CS 188: Artificial Intelligence
Spring 2010

Lecture 17: Bayes' Nets IV – Inference

3/16/2010

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Many slides over this course adapted from Dan Klein, Stuart Russell, Andrew Moore

Announcements

- Assignments
  - W4 back today in lecture
  - Any assignments you have not picked up yet
    - In bin in 283 Soda [same room as for submission drop-off]
- Midterm
  - 3/18, 6-9pm, 0010 Evans --- no lecture on Thursday
  - We have posted practice midterms (and finals)
  - One note letter-size note sheet (two sides), non-programmable calculators [strongly encouraged to compose your own!]
  - Topics go through last Thursday
  - **Section this week: midterm review**

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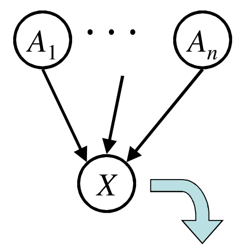
Bayes' Net Semantics

- Let's formalize the semantics of a Bayes' net
- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

    $$P(X|a_1 \dots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process



$$P(X|A_1 \dots A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

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Probabilities in BNs

- For all joint distributions, we have (chain rule):

$$P(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} P(x_i|x_1, \dots, x_{i-1})$$

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} P(x_i|\text{parents}(X_i))$$

- This lets us reconstruct any entry of the full joint
- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

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Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

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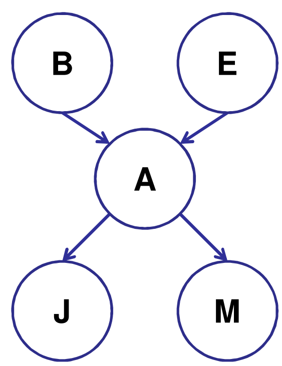
Inference

- Inference: calculating some useful quantity from a joint probability distribution
- Examples:
  - Posterior probability:

$$P(Q|E_1 = e_1, \ldots E_k = e_k)$$

  - Most likely explanation:

$$\text{argmax}_q \; P(Q = q|E_1 = e_1 \ldots)$$



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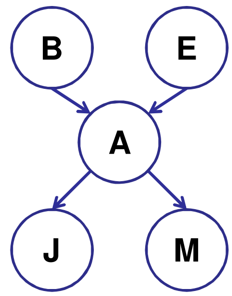
Inference by Enumeration

- Given unlimited time, inference in BNs is easy
- Recipe:
  - State the marginal probabilities you need
  - Figure out ALL the atomic probabilities you need
  - Calculate and combine them
- Example:

$$P(+b|+j,+m) = \frac{P(+b,+j,+m)}{P(+j,+m)}$$



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Example: Enumeration

- In this simple method, we only need the BN to synthesize the joint entries

$$\begin{aligned}P(+b,+j,+m) = \\ & P(+b)P(+e)P(+a|+b,+e)P(+j|+a)P(+m|+a)+ \\ & P(+b)P(+e)P(-a|+b,+e)P(+j|-a)P(+m|-a)+ \\ & P(+b)P(-e)P(+a|+b,-e)P(+j|+a)P(+m|+a)+ \\ & P(+b)P(-e)P(-a|+b,-e)P(+j|-a)P(+m|-a)\end{aligned}$$

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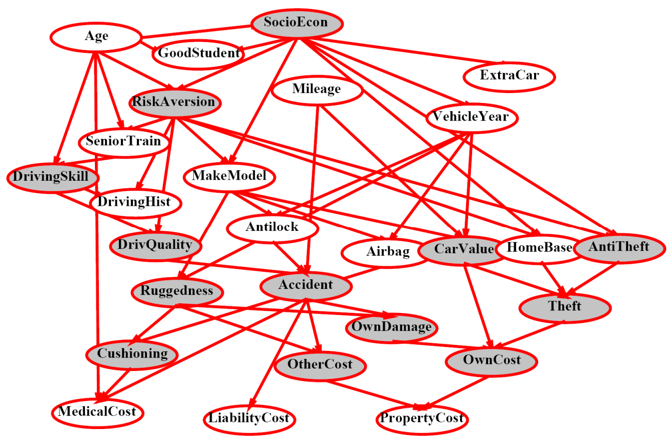
Inference by Enumeration?



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Variable Elimination

- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables
  - You end up repeating a lot of work!
- Idea: interleave joining and marginalizing!
  - Called "Variable Elimination"
  - Still NP-hard, but usually much faster than inference by enumeration
- We'll need some new notation to define VE

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Factor Zoo I

$P(T, W)$

f(W,T)

- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - Sums to 1

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

factor over T and W

- Selected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for fixed x, all y
  - Sums to P(x)

$P(cold, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

factor over cold and W

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Factor Zoo II

P(W|T=cold)

P(W|T=hot)

- Family of conditionals:
  P(X |Y)
  - Multiple conditionals
  - Entries P(x | y) for all x, y
  - Sums to |Y|

$P(W|T)$

factor over W and T

f(W,T)

T	W	P
hot	sun	0.8
hot	rain	0.2
cold	sun	0.4
cold	rain	0.6

$P(W|hot)$

$P(W|cold)$

- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, all y
  - Sums to 1

$P(W|cold)$

f(W, cold)

T	W	P
cold	sun	0.4
cold	rain	0.6

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Factor Zoo III

$p(W|T)$

- Specified family: P(y | X)
  - Entries P(y | x) for fixed y, but for all x
  - Sums to … who knows!

$P(rain|T)$

$f(rain, T)$

T	W	P
hot	rain	0.2
cold	rain	0.6

$P(rain|hot)$
$P(rain|cold)$

- In general, when we write $P(Y_1 \ldots Y_N \mid X_1 \ldots X_M)$
  - It is a "factor," a multi-dimensional array
  - Its values are all $P(y_1 \ldots y_N \mid x_1 \ldots x_M)$
  - Any assigned X or Y is a dimension missing (selected) from the array

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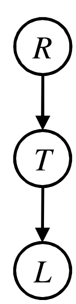
Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!



$P(R)$ $f(R)$

+r	0.1
-r	0.9

$P(T|R)$ $f(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|\cancel{R}T)$ $f(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Joins

Summations

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Variable Elimination Outline

- Track objects called factors
- Initial factors are local CPTs (one per node)

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

- Any known values are selected
  - E.g. if we know $L = +\ell$, the initial factors are

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(+\ell|T)$

+t	+l	0.3
-t	+l	0.1

- VE: Alternately join factors and eliminate variables

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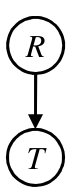
Operation 1: Join Factors

- First basic operation: joining factors
- Combining factors:
  - Just like a database join
  - Get all factors over the joining variable
  - Build a new factor over the union of the variables involved
- Example: Join on R



$P(R) \times P(T|R) \rightarrow P(R,T)$

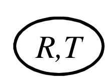
+r	0.1
-r	0.9

1 entry per value for R

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

2 entries per value for R

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81



2 entries per value for R in resulting table

- Computation for each entry: pointwise products

$$\forall r, t : \quad P(r,t) = P(r) \cdot P(t|r)$$

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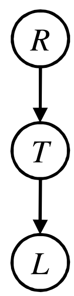
Example: Multiple Joins



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join R

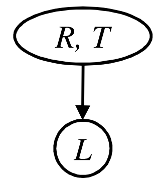
$P(R, T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

factor(R,T)



L|T

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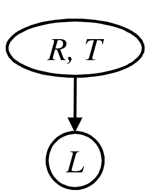
Example: Multiple Joins



f(R,T)

join over T

$P(R, T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

→ 2

Join T

$P(L|T)$ f(L,T)

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

→ 2

T, 

+t +r +l 0.024

$P(R, T, L)$ 0.056

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

4 entries / value of t

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Operation 2: Eliminate

- Second basic operation: marginalization
- Take a factor and sum out a variable
  - Shrinks a factor to a smaller one
  - A projection operation
- Example: f(R,T) f(T)

$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

sum R

$P(T)$

+t	0.17
-t	0.83

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Multiple Elimination

join R, join T

R, T, L   *T, L*   *L*

$P(R,T,L)$

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

2^3

Sum out R

f(T,L)

$P(T,L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Sum out T

$P(L)$

+l	0.134
-l	0.886

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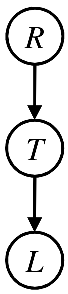
P(L) : Marginalizing Early!



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

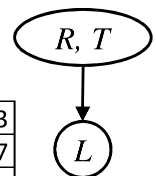
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join R

$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81



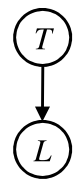
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Sum out R

$P(T)$

+t	0.17
-t	0.83



$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

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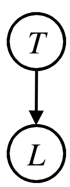
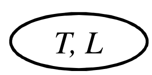
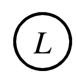
Marginalizing Early (aka VE*)



Join T



Sum out T



$P(T)$

+t	0.17
-t	0.83

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

$P(T,L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

$P(L)$

+l	0.134
-l	0.886

f(L)

* VE is variable elimination

Evidence

$R \rightarrow T \rightarrow L$

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

  - Computing $P(L|+r)$, the initial factors become:

join and sum over T

$P(+r)$

+r	0.1

$P(T|+r)$

+r	+t	0.8
+r	-t	0.2

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

join over T
sum over T
join w/ P(+r)

- We eliminate all vars other than query + evidence

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Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we'd end up with:

$P(+r, L)$

+r	+l	0.026
+r	-l	0.074

Normalize →

$P(L|+r)$

+l	0.26
-l	0.74

- To get our answer, just normalize this!

- That's it!

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P(q)? $\sum_{a,b} \underbrace{P(A)\,P(B|A)\,P(C|B)}$ = $\sum_b P(C=c|B=b) \underbrace{\sum_a P(A=a)\,P(B=b|A=a)}_{f(B)}$

General Variable Elimination

- Query: $P(\underline{Q}|\underbrace{E_1 = e_1, \ldots E_k = e_k}) \quad \sum_a P(A=a) \quad \underbrace{\sum_b P(B=b|A=a)\,P(C=c|B=b)}_{f(B,A,C)}$
- Start with initial factors:
  - Local CPTs (but instantiated by evidence)
- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H

  $f(X_1,H) \; f(X_2,H) \; f(X_3,H)$

  $f(X_1,X_2,X_3,H)$

  $f(X_1,X_2,X_3)$

  $f_1(Q, e_i, e_j, \ldots) \quad f_2(Q, e_k, e_l, \ldots)$
- Join all remaining factors and normalize

  $f(Q, e_1, \ldots, e_l) = P(Q, e_1, \ldots, e_l)$

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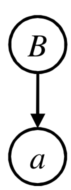
Variable Elimination Bayes Rule

Start / Select

$P(B)$

B	P
+b	0.1
¬b	0.9

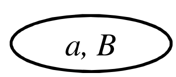


$P(A|B) \rightarrow \boldsymbol{P(a|B)}$

B	A	P
+b	+a	0.8
~~b~~	~~¬a~~	~~0.2~~
¬b	+a	0.1
~~¬b~~	~~¬a~~	~~0.9~~

P(B|a)

Join on B



$P(a, B)$

A	B	P
+a	+b	0.08
+a	¬b	0.09

Normalize

$P(B|a)$

A	B	P
+a	+b	8/17
+a	¬b	9/17

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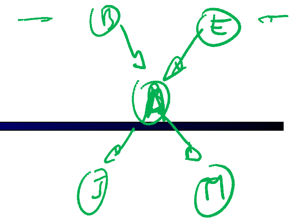
Example



$P(\underline{B}|j,m) \propto \underbrace{P(B,j,m)}$

$P(B)$	$P(E)$	$P(A\|B,E)$	$P(j\|A)$	$P(m\|A)$

2 hidden variables : A, E , 1 query variable : B , 2 evidence vars j, m

Choose A

$$\left.\begin{matrix} P(A|B,E) \\ P(j|A) \\ P(m|A) \end{matrix}\right\} \xrightarrow{\times} \underbrace{P(j,m,A|B,E)}_{f(j,m,A,B,E)} \xrightarrow{\Sigma} \underbrace{P(j,m|B,E)}_{f(j,m,B,E)}$$

$P(B)$	$P(E)$	$P(j,m\|B,E)$

$P(B)\sum_E P(E)\underbrace{\sum_A \underbrace{P(A|B,E)P(j|A)P(m|A)}_{f(B,E,j,m)}}_{f(B,j,m)} = P(B|j,m)$

Example

$P(B)$	$P(\cancel{E})$	$P(j,m\|\cancel{B},E)$

Choose E

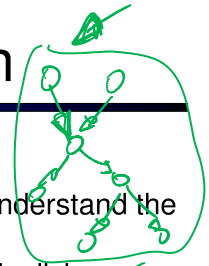
$$\begin{matrix} P(E) \\ P(j,m|B,E) \end{matrix} \xrightarrow{\times} \underbrace{P(j,m,E|B)}_{f(j,m,E,B)} \xrightarrow[E]{\Sigma} \underbrace{P(j,m|B)}_{f(j,m,B)}$$

$P(B)$	$P(j,m\|B)$

Finish with B

$$\begin{matrix} P(B) \\ P(j,m|B) \end{matrix} \xrightarrow{\times} P(j,m,B) \xrightarrow{\text{Normalize}} P(B|j,m)$$

Variable Elimination



- What you need to know:
  - Should be able to run it on small examples, understand the factor creation / reduction flow
  - Better than enumeration: saves time by marginalizing variables as soon as possible rather than at the end

$X_1 \to \circ \to \circ \cdots \circ \to \circ \to X_n$? what factors get generated depending on elimination ordering

- We will see special cases of VE later
  - On tree-structured graphs, variable elimination runs in polynomial time, like tree-structured CSPs
  - You'll have to implement a tree-structured special case to track invisible ghosts (Project 4)

2^n

$P(X_n \mid X_1 = x_1)$

$X_2, X_3, X_4, \ldots, X_{n-1}$ —

$X_{n-1}, X_{n-2}, \ldots, X_2$ —

$X_{n/2}, X_{n/4}$